Stability of trajectories for N -particles dynamics with singular potential

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Abstract

We study the stability in finite times of the trajectories of interacting particles. Our aim is to show that in average and *uniformly in the number of particles*, two trajectories whose initial positions in phase space are close, remain close enough at later times. For potential less singular than the classical electrostatic kernel, we are able to prove such a result, for initial positions/velocities distributed according to the Gibbs equilibrium of the system.

1 Introduction

The stability of solutions to a differential system of the type

$$\frac{dZ}{dt} = F(Z(t)), \tag{1.1}$$

is an obvious and important question. For times of order 1 and if F is regular enough (for instance uniformly Lipschitz), the answer is given quite simply by Gronwall lemma. For two solutions Z and Z^δ to (1.1), one has

$$|Z(t) - Z^\delta(t)| \le |Z(0) - Z^\delta(0)| \, \exp(t \, \|\nabla F\|_{L^\infty}). \tag{1.2}$$

This inequality forms the basis of the classical Cauchy-Lipschitz theory for the well posedness of (1.1). It does not depend on the dimension of the system (the norm chosen is then of course crucial). This is hence very convenient for the study of systems of interacting particles, which is our purpose here.

Consider the system of equations

$$\begin{cases} \dot{X}_i^N = V_i^N \\ \dot{V}_i^N = E_N(X_i^N) = \frac{1}{N}\sum_j K(X_i^N - X_j^N) \end{cases} \tag{1.3}$$

where for simplicity all positions X_i^N belong to the torus $\mathbb{T}^3$ and all velocities V_i^N belong to $\mathbb{R}^3$. This system is obviously a particular case of (1.1) with $Z = Z^N = (X_1^N, \dots, X_N^N, V_1^N, \dots, V_N^N)$.

The equivalent of (1.2) reads in this case

$$\|Z^N(t) - Z^{N,\delta}(t)\|_1 \le \|Z^N(0) - Z^{N,\delta}(0)\|_1 \, \exp(t\,(1 + \|\nabla K\|_{L^\infty})), \tag{1.4}$$

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where we define the norm on $\Pi^{3N}\times\mathbb{R}^{3N}$

$$\|Z\|_1=\frac{1}{2N}\sum_{i=1}^{N}(|X_i|+|V_i|).$$

This estimate is logically uniform in the number of particles N. It is important in itself but also because it is a crucial tool to pass to the limit in the system of N particles and derive the Vlasov-type equation

$$\partial_t f+v\cdot\nabla_x f+\left(F\star_x\left(\int_{\mathbb{R}^3} f(t,x,v)\,dv\right)\right)\cdot\nabla_v f=0, \tag{1.5}$$

for the 1-particle density $f(t,x,v)$ in phase space, where $\star$ denotes the convolution. Hence estimates such as (1.4) are at the heart of the derivation performed in [20], [10], and [4] (we also refer to [23], [24] and [26]). Note that the derivation of collisional kinetic models (of Boltzmann type) involves quite different techniques, see [18] or [5].

Unfortunately, many cases of interest in physics deal with singular forces $K\notin W^{1,\infty}_{loc}$. Typical cases are $K=-\nabla\phi$, a periodic force coming from a periodisation of the potential $\phi_{\mathbb{R}}(x)\sim C/|x|^{\alpha-1}$, *i.e.* $\phi(x)=\phi_{\mathbb{R}}(x)+g(x)$, where $g(x)$ is an (at least) C^2 function on the torus $\mathbb{T}^3$. As the potential ϕ is defined up to a constant, we may also assume that its average is 0: $\int_{\mathbb{T}^3}\phi(x)\,dx=0$. The most important case is the electrostatic or gravitational interaction: $\alpha=2$, in dimension 3.

Very little is known for these singular kernels, either from the point of view of the stability or of the derivation of Vlasov-type equations. Provided $\alpha<1$ and the initial configuration of particles are well distributed, the limit to Vlasov equation (1.5) was proved in [14].

For systems without inertia, *i.e.* when the equations are simply

$$\left\{\ \dot X_i^N=E_N(X_i^N)=\tfrac{1}{N}\sum_j K(X_i^N-X_j^N)\right. \ , \tag{1.6}$$

it seems to be easier to implement Gronwall-type inequalities. The derivation of the mean field limit is consequently known up to $\alpha<2$ ($\alpha<d-1$ in dimension d), see [13] and also [17] for a situation where the forces have a more complicated structure. In this setting the most important case is however found in dimension 2, for $K=x^\perp/|x|^2$ (corresponding to $\alpha=d-1=1$); the limit is the 2d incompressible Euler equation written in vorticity form. The derivation of the mean-field limit in this case was rigorously performed in [11] and [21], [22].

For differential equations like (1.1) in finite dimensions, it has long been known that well posedness and stability (for almost all initial data) can be achieved without using Gronwall-type estimates. The introduction of renormalized solutions by DiPerna-Lions in [9] gave well posedness for $F\in W^{1,1}$ with $div\,F\in L^\infty$.

This was extended to $F\in BV$ in the phase space situation in [3] and then in the general case in [1] (see also [15] for a slightly different approach). The exact case of the Poisson interaction was treated in [12].

This well posedness implies some stability as the flow has then some differentiability properties, see [2]. However the corresponding stability estimate is not quantitative and this kind of method does not seem to be able to provide uniform estimates in the number of particles (which gives the dimension of the system). We refer to [8] for a precise overall presentation of the well posedness and differentiability issues for Eq. (1.1) in finite dimension.

More recently a new method to show well posedness for (1.1) has been introduced in [7]. Given a fixed shift δ, it consists in bounding quantities like

$$\int_{Z^0}\log\left(1+\frac{|Z(t,Z^0)-Z(t,Z^0+\delta)|}{|\delta|}\right)dZ^0, \tag{1.7}$$

where Z is the flow associated to (1.1), *i.e.* $Z(t,Z^0)$ is a solution to (1.1) satisfying $Z(0,Z^0)=Z^0$.

A bound on such a quantity shows that for *a.e.* Z^0 the two trajectories $Z(t,Z^0)$ and $Z(t,Z^0+\delta)$ remain at a distance of order $|\delta|=|Z(0,Z^0)-Z(0,Z^0+\delta)|$. In this sense, this is an almost-everywhere version of the Gronwall inequality (1.2).

It was shown in [7] that the quantity (1.7) remains bounded if $F \in W^{1,p}$ for some $p > 1$. This was extended to $W^{1,1}$ and SBV in [16] and even to $H^{3/4}$ in the phase space setting in [6]. However in all those results the bound depends on the dimension of the space and is blowing-up as this dimension increases to ∞.

Therefore, our precise aim in this article is to prove a bound on a quantity like (1.7) for the system (1.3), uniformly in the number of particles. To our knowledge, this is the first quantitative stability estimate to be obtained for singular forces.

Several new important issues occur when one tries to do that though. One of the most important is the reference measure which is chosen as this can imply different notions of almost everywhere as the dimension tends to $+\infty$. In finite dimension, this refers to the Lebesgue measure and of course implies corresponding estimates for any measure which is absolutely continuous with respect to the Lebesgue measure. In infinite dimension, no such natural measure exists. This is due to the phenomenon of concentration of measures. Even in the case of finite but large dimensions, this is a problem to get quantitative estimates. Indeed even if two measures ν_1 and ν_2 on $\Pi^{3N} \times \mathbb{R}^{3N}$ are absolutely continuous with respect to each other or even more if

$$d\nu_1 \leq C\, d\nu_2,$$

then the constant C will in general depend on the dimension and go to $+\infty$ as N increases. This means that a uniform quantitative estimate on the trajectories for some measure $d\nu_1(Z^0)$ on the initial configuration will not in general imply a good estimate for another measure $d\nu_2(Z^0)$.

For each N, the choice of the measure μ_N on $\Pi^{3N} \times \mathbb{R}^{3N}$ is hence crucial to get a good estimate. One would naturally want to select a measure μ_N which is bounded, stable and invariant by the flow, just as the Lebesgue measure is stable and invariant by the flow of (1.1) when F is divergence free. Let us denote by $Z^N(t) = (X^N(t), V^N(t))$ or $Z^N(t, Z_0^N)$ the vector of particles velocities and positions evolving through Eq. (1.3) till time t and depending on the initial configuration Z_0^N. The system (1.3) has an invariant which is the total energy

$$\begin{aligned} H_N[Z^N] &= \sum_i \frac{(V_i^N)^2}{2} + \frac{1}{2N}\sum_{i\neq j} \phi(|X_i^N - X_j^N|) && (1.8) \\ &= E_{kin}(V^N) + E_{pot}(X^N) && (1.9) \end{aligned}$$

To get an invariant measure μ_N, the simplest choice is to take a function of the total energy. Among those which are stable, the most natural is the Gibbs equilibrium

$$d\mu_N(Z^N) = \frac{1}{\mathcal{B}_N} e^{-\beta H_N[Z^N]} dZ^N \ , \qquad (1.10)$$

where dZ^N is Lebesgue measure on $\mathbb{T}^{3N} \times \mathbb{R}^{3N}$, and

$$\mathcal{B}_N(\beta) = \int e^{-\beta H_N[Z^N]}\, dZ^N \ , \qquad (1.11)$$

is the normalization constant. Note that for a potential ϕ with a singularity at the origin, this makes sense only if $\lim_0 \phi = +\infty$, that is in the case of repulsive interactions. In the following, since we deal with measures which have a density with respect to the Lebesgue measure dZ_0^N, we will use the same notation for the measure μ_N and its density.

We study the quantity

$$Q(t) = \int d\mu_N(Z_0^N) \int_{\delta \in \mathbb{T}^{3N}\times\mathbb{R}^{3N}} \psi_N(Z_0^N, \delta) \ln\left(1 + \frac{\|Z^N(t, Z_0^N) - Z^N(t, Z_0^N + \delta)\|_1}{\delta_N}\right) d\delta \ , \qquad (1.12)$$

where δ_N is a small parameter that will go slowly to zero when N goes to infinity. δ is a shift on the initial condition Z_0^N

Here $\psi_N : \mathbb{T}^{3N} \times \mathbb{R}^{3N} \mapsto \mathcal{P}(\mathbb{T}^{3N} \times \mathbb{R}^{3N})$ (where $\mathcal{P}(\Omega)$ denotes the set of probabilities on Ω) is a probability valued function, so that it satisfies

$$\int_{\delta \in \mathbb{T}^{3N} \times \mathbb{R}^{3N}} d\psi_N(Z_0^N, \delta) = 1, \quad \forall Z_0^N \in (\mathbb{T}^3 \times \mathbb{R}^3)^N$$

ψ_N then then gives distribution of the shifts on the initial conditions, and the quantity $Q(t)$ is averaged both on the initial conditions Z_0^N and on the shifts δ.

We now define the image measure of μ_N by the shift distribution:

$$\tilde{\mu}_N(Z_0) = \int \mu_N(Z_0 - \delta)\psi_N(Z_0 - \delta, \delta)\, d\delta\,.$$

The crucial assumption will be that the image measure $\tilde{\mu}_N$ remains "close" to the original Gibbs measure in the sense that

$$\exists\, K_\beta > 0, \text{ such that } \forall\, Z_0 \text{ and } N, \quad \tilde{\mu}_N(Z_0) \le K_\beta \mu_N(Z_0) \tag{1.13}$$

with a constant K_β independent of N, but which may depend on β.

We will also use the weaker but very similar condition

$$\exists\, K'_\beta > 0 \text{ s.t. } \forall\, N, \exists\, \beta' \le \beta, \text{ s.t. } \forall\, Z_0,\ \tilde{\mu}_N(Z_0) \le K'_\beta \mu_N^{\beta'}(Z_0), \tag{1.14}$$

where $\mu_N^{\beta'}$ denotes the Gibbs measure with inverse temperature β. The last condition is more general than the first, it allows to control the image measure by a Gibbs measure with bigger temperature.

Remark 1.1 *We mention that by definition of $\tilde{\mu}_N$, the measure $\pi_N(Z_0, Z'_0) = \mu_N(Z_0)\Phi_N(Z_0, Z'_0 - Z_0)$ is a transport from the measure μ_N to $\tilde{\mu}_N$. In fact, ψ_N is (up to a translation of origin) what is usually called the desintegration of the measure π_N with respect to its first projection μ_N. However, we preferred our less standard presentation since we are more interested in μ_N and its shift ψ_N that in the precise image measure $\tilde{\mu}_N$. We mention the analogy to emphasize that our quantity Q is related to optimal transport. In fact*

$$Q_N(0) = \int d\mu_N(Z_0^N) \int_{\delta \in \mathbb{T}^{3N} \times \mathbb{R}^{3N}} d\psi_N(Z_0^N, \delta) \ln\left(1 + \frac{\|\delta\|_1}{\delta_N}\right) \ge W_N(\mu_N, \tilde{\mu}_N)$$

where W_N is the transport for the cost $\ln\left(1 + \frac{\|\cdot\|_1}{\delta_N}\right)$.

Conditions (1.13) and (1.14) are not explicit on ψ. Roughly speaking, they will be satisfied if ψ_N is chosen so that $|H_N(Z_0 + \delta) - H_N(Z_0)| \le C$ if $\delta \in \operatorname{Supp} \Psi_N(Z_0, \cdot)$. This is reasonable since H_N is preserved by the dynamics, so that if the shift changes the energy too much, the original and shifted dynamics may be very different. But that simple and "reasonable" condition is not sufficient, we will really need a bound like (1.13) on the image measure constructed with the shift.

As the conditions are not explicit, we will provide in section 2 some examples of admissible shift distributions. The main result of the paper is a control on the growth on this quantity Q:

Theorem 1.2 *Assume that $\phi \ge \phi_{min}$ for some $\phi_{min} \in \mathbb{R}^-$ and that for some constant C, and $\alpha < 2$*

$$\phi(x) \le \frac{C}{|x|^{\alpha-1}}, \quad |\nabla \phi| \le \frac{C}{|x|^\alpha}, \quad |\nabla^2 \phi| \le \frac{C}{|x|^{\alpha+1}}.$$

Then taking $\delta_N = N^{-\varepsilon}$ for any $\varepsilon \le 1 - \alpha/3$ and for all $N \ge \frac{6^4}{(2-\alpha)^2}$ one has

$$Q(t) \le \left(1 + (1 + K_\beta)\frac{Cc_\beta + C_a c_\beta^a}{2 - \alpha}\right) t + Q(0)\ ,$$

where $c_\beta = e^{-\frac{\beta}{2}\phi_{min}}$, a is any exponent strictly larger than $2\alpha/3$, C constant (that can be made explicit), and C_a satisfies $C_a \le \frac{C}{3a-2\alpha}$.

This theorem is not able to deal with the electrostatic interaction, $\alpha = 2$; gravitational is of course out of question since repulsive potentials are needed. Note however that the electrostatic potential is just the critical case. The same result could be obtained in any dimension, with essentially the same proof. In dimension d, the condition would then be $\alpha < d - 1$. The growth of Q is linear in time: note that this indeed corresponds to an average exponential in time divergence of the trajectories, analogous to (1.2).

Roughly speaking, and provided that the average shift at time 0 is of order δ_N (or smaller), the theorem says that the average shift transported by the dynamics remains of order δ_N during the evolution, and the control given is quite good. It is interesting to compare δ_N to the minimal distance in the (X, V) space between two particles of a configuration, which is of order $N^{-\frac{1}{3}}$. Notice that it is always possible to choose δ_N smaller than $N^{-\frac{1}{3}}$. Then if the order of magnitude of the initial shifts is smaller than $N^{-\frac{1}{3}}$, the theorem says it remains so at all time. This implies that the pairing of a particle in the configuration $Z(t)$ with the closest one in the configuration $Z^\delta(t)$ is not very much affected by the dynamics: in this sense, there is not much "mixing".

While the Gibbs equilibrium is the most natural choice for the measure μ_N, others are possible. The proof would work for any measure μ_N such that

- μ_N is invariant under the flow or $\mu_N(Z^N(t)) = \mu_N(Z_0^N)$ for Z^N solution to (1.3)
- for all k, the k-marginal of μ_N defined by $\mu_N^k(Z^k) = \int \mu_N(Z_0^k, \tilde{Z}_0^{N-k}) d\tilde{Z}_0^{N-k}$ satisfies:

$$\forall Z_0, \quad \mu_N^k(Z_0) \leq C^k.$$

Obvious candidates are functions of the renormalized energy $\frac{1}{N}H_N$ but checking the bound on the marginals is not necessarily easy.

Link with mean field limit.

Finally, let us emphasize that this stability estimate does not answer the question of the mean field limit. Doing so would require to be able to deal with much more general measures μ_N. More precisely if one can prove Th. 1.2 for a sequence of μ_N and if in some reasonable sense

$$\mu_N - \Pi_{i=1}^N f^0 \longrightarrow 0,$$

then the mean field limit is proved but only for the initial data f^0. Currently the Gibbs equilibrium corresponds to $f^0(x, v) = e^{-\beta|v|^2/2}$.

Unfortunately, we are not able do deals with more general measures μ_N^0. The problem is that we need bounds on every k marginals and those are very difficult to obtain if we start from another measure than the Gibbs equilibrium. For instance, starting from $\mu_N = g^{\otimes N}$ for some smooth profil g, we have the desired bounds at time 0, but do not know if $\mu_N(t)$ satisfies them for any other time $t > 0$.

2 Some examples of admissible shift distributions.

2.1 Shift on velocity variables

In this section, we will be interested in shifts acting only the velocities. A first possibility is to take shifts independent of Z_0 and acting independently and identically on each velocity variable. Precisely, we are looking for shift distributions such as

$$\psi_N(\delta) = \delta_{\delta_X = 0} \prod_{i=1}^N N^{\frac{3}{2}} \psi(\sqrt{N}\delta_{v_i})$$

where ψ is a probability on $\mathbb{R}^3$ symmetric with respect to the origin ($\psi(-v) = \psi(v)$ if ψ has a density).

We will not be able to deal with a general ψ, but will show that the hypothesis (1.14) is satisfied if ψ is Gaussian or has a compact support. This is stated precisely in the following Proposition:

Proposition 2.1 *Assume that ψ is a Gaussian probability with variance σ^2 :*

$$\psi(\delta_v) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{3}{2}} e^{-\frac{\delta_v^2}{2\sigma^2}}$$

then the hypothesis (1.14) *is satisfied with*

$$\beta'(N) = \beta\left(1 - \frac{1}{1+N/(\beta\sigma^2)}\right) , \text{ and } K_\beta = e^{-\frac{\beta^2\sigma^2}{4}\phi_{min}} e^{\frac{3}{4}\beta\sigma^2} .$$

Assume that ψ has a compact support with $\mathrm{Supp}\,\psi \subset B(0,\delta_m)$ *(the ball of radius δ_m, center* 0*). Then* (1.14) *is satisfied for $N > \beta\delta_m^2$ with*

$$\beta'(N) = \beta\left(1 - \frac{\beta\delta_m^2}{N}\right) , \text{ and } K_\beta = e^{-\frac{\beta^2\delta_m^2\phi_{min}}{2}} e^{\frac{3}{2}\beta\sigma^2}$$

Remark that for $\alpha \leq \frac{3}{2}$, the average velocity fluctuation given by such shift is larger than the smallest δ_N we can choose.

Proof of the proposition 2.1
In the Gaussian case, we have

$$\tilde{\mu}_N(Z_0) = \frac{e^{-\beta H_N(Z_0)}}{\mathcal{B}_N(\beta)} \left(\frac{N}{2\pi\sigma^2}\right)^{\frac{3N}{2}} \int e^{-\frac{\beta}{2}\sum_i(\delta_{v_i}^2 - 2v_i\delta_{v_i})} e^{-\frac{N}{2\sigma^2}\sum_i \delta_{v_i}^2}\, d\delta_V .$$

The $3N$ integrals may be performed independently, using the 1D calculation:

$$\sqrt{\frac{N}{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(\beta+N/\sigma^2)\delta_v^2 + \beta v\delta_v}\, d\delta_v = \sqrt{\frac{N}{N+\beta\sigma^2}} e^{\frac{\beta}{2}\frac{\beta\sigma^2}{N+\beta\sigma^2}}$$

We finally get

$$\begin{aligned} \tilde{\mu}_N(Z_0) &= \frac{e^{-\beta H_N(Z_0)}}{\mathcal{B}_N(\beta)} \left(\frac{1}{1+\beta\sigma^2/N}\right)^{\frac{3N}{2}} e^{\frac{\beta}{1+N/\beta\sigma^2}E_{kin}(Z_0)} \\ &= \frac{e^{-\beta' H_N(Z_0)}}{\mathcal{B}_N(\beta')} \frac{\mathcal{B}_N(\beta')}{\mathcal{B}_N(\beta)} \left(\frac{1}{1+\beta\sigma^2/N}\right)^{\frac{3N}{2}} e^{-\frac{\beta}{1+N/\beta\sigma^2}E_{pot}(Z_0)} \end{aligned}$$

with $\beta'(N) = \beta(1 - \frac{1}{1+N/(\beta\sigma^2)})$. Using

$$\frac{\beta}{1+N/\beta\sigma^2} E_{pot}(Z_0) \geq \frac{\beta^2\sigma^2\phi_{min}}{4} ,$$

we get

$$\tilde{\mu}_N(Z_0) \leq e^{-\frac{\beta^2\sigma^2}{4}\phi_{min}} e^{-\frac{3}{4}\beta\sigma^2} \frac{\mathcal{B}_N(\beta')}{\mathcal{B}_N(\beta)} \tilde{\mu}_N^{\beta'}(Z_0)$$

We define $\mathcal{B}_{N,X}$ the normalization integral restricted to the position variables:

$$\mathcal{B}_{N,X}(\beta) = \int e^{-\beta E_{pot}(X_1,\ldots,X_N)}\, dX_1 \ldots dX_N$$

From the Jensen inequality applied to the function $x \mapsto x^{\beta'/\beta}$, we obtain for $\beta' \leq \beta$

$$(\mathcal{B}_{N,X}(\beta')) \leq (\mathcal{B}_{N,X}(\beta))^{\beta'/\beta}$$

Then, using the bounds of Lemma 3.1, we have $\mathcal{B}_{N,X}(\beta)^{\beta'/\beta-1} \leq 1$; this implies

$$\frac{\mathcal{B}_N(\beta')}{\mathcal{B}_N(\beta)} \leq \left(\frac{\beta}{\beta'}\right)^{3N/2} \leq e^{3\beta\sigma^2/2}$$

This proves the desired inequality with

$$K_\beta = e^{-\frac{\beta^2\sigma^2}{4}\phi_{min}} e^{\frac{3}{4}\beta\sigma^2} .$$

In the case of ψ with compact support, we follow the same sketch. To do this, we will need a bound on

$$\int e^{\frac{\beta}{2}(2v\delta_v - \delta_v^2)} N^{\frac{3}{2}}\, d\psi(\sqrt{N}\delta_v) = \int e^{\frac{\beta}{2}\left(\frac{2v\delta_v}{\sqrt{N}} - \frac{\delta_v^2}{N}\right)}\, d\psi(\delta_v) .$$

Using the symmetry of ψ, and the inequality $\cosh(x) \leq e^{\frac{x^2}{2}}$ (valid for $x \in \mathbb{R}$), we may bound that term by

$$\int \cosh\left(\frac{\beta v\delta_v}{\sqrt{N}}\right) e^{-\frac{\beta\delta_v^2}{2N}}\, d\psi(\delta_v) \leq \int e^{\frac{\beta^2 v^2\delta_v^2}{2N} - \frac{\beta\delta_v^2}{2N}}\, d\psi(\delta_v) \leq e^{\frac{\beta^2\delta_m^2 v^2}{2N}}$$

and as in the previous case, we get

$$\int \mu_N(Z_0 - \delta)\psi_N(Z_0 - \delta, \delta)\, d\delta \leq \frac{1}{\mathcal{B}_N} e^{-\beta\left(H_N(Z_0) - \frac{\beta\delta_m^2}{N} E_{kin}(Z_0)\right)} .$$

From this point, following exactly the same step as in the case of gaussian ψ, we prove that the hypothesis (1.14) is satisfied with the announced constant $\square$

.

By making the shift depend on the velocity $V_0 = (V_1, \dots V_N)$, one may essentially remove the condition on the size of the norm of the shift. More precisely we limit ourselves to shifts $\delta = (0, \dots, 0, \delta')$ with $\delta' \in \mathbb{R}^{3N}$, giving $Z_0 + \delta = (X_0, V_0 + \delta')$ with $X_0 = (X_1, \dots, X_N)$. Now define

$$\psi_N(Z_0, \delta) = \left(\Pi_{i=1}^N \delta_{\delta_i = 0}\right)\ \Psi(|\delta'|)\, G(|V_0|)\, \delta_{2\, V_0\cdot\delta' + |\delta'|^2 = 0}, \tag{2.1}$$

where $|\xi|^2 = \xi \cdot \xi$ is the usual euclidian norm and $\delta_{\dots}$ is the corresponding Dirac mass on the hypersurface of equation $2\, V_0 \cdot \delta' + |\delta'|^2 = |V_0 + \delta'|^2 - |V_0|^2 = 0$ which is precisely the sphere of radius $|V_0|$.

We need the function ψ_N to satisfy

$$1 = \int \psi_N(Z_0, \delta)\, d\delta = G(|V_0|) \int_{2\, V_0\cdot\delta' + |\delta'|^2 = 0} \Psi(|\delta'|)\, d\delta',$$

this is always possible with the right choice of G as the integral

$$\int_{2\, V_0\cdot\delta' + |\delta'|^2 = 0} \Psi(|\delta'|)\, d\delta'$$

depends only on $|V_0|$ by the rotational symmetry of the sphere.

Condition (1.13) is automatically true since, as μ_N depends only on X_0 and $|V_0|$ and $|V_0 - \delta'| = |V_0|$,

$$\tilde{\mu}_N(Z_0) = \int \mu_N(Z_0 - \delta)\, \psi_N(Z_0 - \delta, \delta) = \mu_N(Z_0) \int \psi_N(Z_0, -\delta) = \mu_N(Z_0),$$

because $\psi_N(Z_0 - \delta, \delta) = \psi_N(Z_0, -\delta)$. One could wish to impose additionally that $|\delta'|_1 \leq \delta_N$, so that $Q(0)$ is of order 1. Since

$$|\delta'|_1 = \frac{1}{N}\sum_i |\delta_i'| \leq \frac{1}{\sqrt{N}}\, |\delta'|,$$

it is enough to impose that Ψ has support in $[0, N^{1/2}\, \delta_N]$.

As a conclusion, we proved

Proposition 2.2 *For any measure Ψ on $\mathbb{R}^{3N}$, there exists a function $G(|V_0|)$, s.t. the probability density ψ_N defined by* (2.1) *satisfies* (1.13).

One could try to generalize this example by letting $V_0 + \delta'$ and V_0 to be on close but different energy spheres. For example by posing

$$\psi_N(Z_0, \delta) = \left(\Pi_{i=1}^{N} \delta_{\delta_i = 0}\right) \; \Psi(|\delta'|) \, G_\eta(|V_0|) \; \mathbb{I}_{|2V_0 \cdot \delta + \delta^2| \leq \eta}.$$

Provided that η in not too large, the computations are essentially the same and one has essentially to make sure that

$$G_\eta(|V_0| \pm \eta) \leq C \, G_\eta(|V_0|).$$

2.2 Shifts in position variables

One could try to implement the same idea for shifts in position variables. Many problems arise however since the potential energy is not a nicely regular function of the positions.

If one tries to consider shift distributions $\psi_N(\delta)$ that do not depend on Z_0, then the limitation on $|\delta|$ is quite drastic. In fact this example essentially works if only a fixed (independent of N) number of coordinates δ_i are not 0.

Trying to generalize the second example by imposing that $Z_0 + \delta$ and Z_0 are on the same energy surface also faces several problems. The main one is that the equation of the energy surface is not anymore symmetric between the shift δ and the shift $-\delta$.

The only solution would be to write the equation only on the tangent plane, *i.e.* something like

$$\psi_N = \Psi(|\delta|) \, \delta_{\nabla H(Z_0) \cdot \delta = 0} \, |\nabla H(Z_0)|.$$

This is now nicely symmetric but poses other difficulties. For instance one would need to make sure that $H(Z_0 + \delta) \leq H(Z_0) + C$. Expanding H, one would formally find a condition of the type

$$|\nabla^2 H(Z_0)| \, |\delta|^2 \leq C.$$

Unfortunately $\nabla^2 H(Z_0)$ is singular and in particular unbounded. It is only bounded in average which would force us to remove the initial conditions around which the energy has a singular behavior.

Although this procedure could in principle be carried out successfully, we do not wish to enter here into such technical computations. This essentially limits us to present explicit shift distributions acting only on velocity variables.

However, let us point out that the evolution of the particles system strongly mixes positions and velocities. Obviously, if we start from two initial conditions Z_0 and Z_0^δ with same positions and different velocities, we get at time $t > 0$ configurations with different positions.

So if we really need a shift distribution that acts also on the positions at the origin $t = 0$, a strategy may be to start at $t = -\tau > 0$, using a shift distribution acting only on velocities at this time, and then to let evolves the particles till time 0. The original shift distribution transported by the flows is now a shift distribution acting on position and speed. Using the Theorem (1.2), we get that the average of these evolved shifts (in a weak sense since we we are taking a logarithmic mean) is at most of order δ_N, provided the average of the original shifts was also of order at most δ_N. Therefore by removing a set of vanishing measure of initial conditions, one obtains a shift distribution in positions and velocities that satisfies all the requirements.

3 Some useful bounds for the $6N$ dimensional μ_N

We shall make use of the following lemmas. Before stating them, we introduce a notation for the projection of μ_N on the position space.

$$\nu_N(X^N) = \int \mu_N(X^N, V^N) \, dV^N \,. \tag{3.1}$$

We also denote the k- marginal of ν_N by ν_N^k.

Lemma 3.1 *For all N, we have:*

$$\left(\frac{2\pi}{\beta}\right)^{3N/2} \leq \mathcal{B}_N \leq \left(\frac{2\pi e^{-\beta\frac{\phi_{min}}{3}}}{\beta}\right)^{3N/2} . \tag{3.2}$$

We will also need the following estimate on the kth marginal of μ_N defined by

$$\mu_N^k(Z^k) = \int \mu_N(dZ^{N-k}) = \frac{1}{\mathcal{B}_N} \int e^{-\beta H_N(Z^k, Z^{N-k})}\, dZ^{N-k} , \tag{3.3}$$

and more precisely on the k marginal ν_N^k of the projection ν_N on the position variables.

Lemma 3.2 *We define the constant $c_\beta = e^{-\beta\phi_{min}}$. Then, for all k, we have*

$$\nu_N^k(X^k) \;\leq\; c_\beta^k \tag{3.4}$$

Proof of lemma 3.1 To compute the integral defining $\mathcal{B}_N$ (1.11), we may separate the integration in X^N from the integration in V^N. In the V^N variable, we have to integrate a product of $3N$ independent real gaussians of variance β^{-1}. We obtain $(2\pi/\beta)^{3N/2}$.

In the X^N variable, we use Jensen inequality by the convexity of exponential to get:

$$\begin{aligned} 1 &= e^{\int \frac{-\beta}{2N}\sum_{i\neq j}\phi(|X_i^N - X_j^N|)\, dX^N} \\ &\leq \int e^{\frac{-\beta}{2N}\sum_{i\neq j}\phi(|X_i^N - X_j^N|)}\, dX^N \\ &= \left(\frac{2\pi}{\beta}\right)^{-3N/2} \int e^{-\frac{\beta}{2}\sum_i |V_i^N|^2 - \frac{\beta}{2N}\sum_{i\neq j}\phi(|X_i^N - X_j^N|)}\, dX^N\, dV^N \\ &= \left(\frac{2\pi}{\beta}\right)^{-3N/2} \mathcal{B}_N, \end{aligned}$$

which gives the first inequality (We used that ϕ has zero average). To obtain the second bound, it suffices to use the inequality $E_{pot}(Z_0^N) \geq \frac{N}{2}\phi_{min}$. $\square$

Proof of lemma 3.2 The proof follows the one introduced in [19] for the Lame-Enden equation. As the measure μ_N factorizes in position and speed, we may write

$$\nu_N(X^N) = \frac{1}{\mathcal{B}_{N,X}} e^{-\beta E_{pot}(X^N)}$$

Neglecting the terms in interaction energy involving (at least) one of the first k particles, we obtain

$$\begin{aligned} \nu_N^k(X^k) &= \frac{1}{\mathcal{B}_{N,X}} \int e^{-\beta E_{pot}(X^k, X^{N-k})}\, dX^{N-k} \\ &\leq \frac{1}{\mathcal{B}_{N,X}} e^{-\beta k\phi_{min}} \int e^{-\beta\frac{N-k}{N}E_{pot}(X^{N-k})}\, dX^{N-k} . \end{aligned} \tag{3.5}$$

the term $\frac{N-k}{N}$ being there because $E_{pot}(X_k) = (1/k)\sum_{i\neq j}^k \phi(|X_i - X_j|)$ for k positions. So we need an estimate on terms of the kind

$$\Theta(k) = \int e^{-\beta\frac{k}{N}E_{pot}(X^k)}\, dX^k ,$$

We can relate this term to configurations with $k+1$ particles. First use Jensen inequality as the exponential is convex to get

$$\begin{aligned}\Theta(k) &= \int e^{-\beta \frac{k}{N} E_{pot}(X^k)}\, dX^k \\ &= \int \exp\left(-\int \left(\beta \frac{k}{N} E_{pot}(X^k) + \frac{\beta}{N}\sum_{i=1}^{k} \phi(|X_i - x_{k+1}|)\right) dx_{k+1}\right) dX^k \\ &\le \int e^{-\beta \frac{k}{N} E_{pot}(X^k) - \frac{\beta}{N}\sum_{i=1}^{k} \phi(|X_i - x_{k+1}|)}\, dX^k\, dx_{k+1} \\ &= \int e^{-\beta \frac{k+1}{N} E_{pot}(X^{k+1})}\, dX^{k+1} \\ &\le \Theta(k+1).\end{aligned}$$

Since, $\Theta(N) = \mathcal{B}_{N,X}$, we iterate this inequality and get $\Theta(N-k) \le \mathcal{B}_{N,X}$. Putting this in (3.5), we get

$$\nu_N^k(X^k) \le e^{-k\beta\phi_{min}}\,,$$

which is the result needed. □

4 Proof of Theorem 1.2

During the course of the demonstration, C will denote a constant (independent of N and β), which value may change from line to line.

From now on, we shall omit the superscript N in the notation Z^N, as there will be no ambiguity. We have to estimate the derivative of $Q(t)$. Differentiating directly, one obtains

$$\frac{d}{dt}Q(t) \le \int d\mu_N(Z_0) \int d\delta\, \psi_N(Z_0,\delta) \left(\frac{\frac{1}{N}\sum_i |V_i - V_i^\delta|}{\delta_N + \|Z - Z^\delta\|_1} + \frac{\frac{1}{N^2}\sum_i |\sum_j (K(X_i - X_j) - K(X_i^\delta - X_j^\delta))|}{\delta_N + \|Z - Z^\delta\|_1} \right),$$

where $Z^\delta = (X^\delta, V^\delta) = Z(t, Z_0 + \delta)$.

Note that the first term is obviously bounded by 1 and hence

$$\frac{d}{dt}Q(t) \le 1 + \int d\mu_N(Z_0) \int d\delta\, \psi_N(Z_0,\delta)\, \frac{\frac{1}{N^2}\sum_i |\sum_j (K(X_i - X_j) - K(X_i^\delta - X_j^\delta))|}{\delta_N + \|Z - Z^\delta\|_1}.$$

We define for a integer L that will be fixed later

$$\mathcal{C}_i(Z_0,t) = \{j \ne i \ s.t.\ |X_i(t) - X_j(t)| \text{ is among the } L \text{ smallest } |X_i - X_k|\} \tag{4.1}$$

That is for each i, $\mathcal{C}_i$ collects the indices of particles which are closest to particle i at time t, following the flow. It also depends on the initial condition Z_0. We define similarly $\mathcal{C}_i^\delta$.

Accordingly, we decompose dQ/dt as follows

$$\frac{d}{dt}Q(t) \le C + S_1 + S_1^\delta + S_2,$$

with

$$\begin{aligned} S_1 &= \int d\mu_N(Z_0) \int d\delta\, \psi_N(Z_0,\delta)\, \frac{1}{\delta_N\, N^2} \sum_i \sum_{j \in \mathcal{C}_i \cup \mathcal{C}_i^\delta} |K(X_i - X_j)|, \\ S_1^\delta &= \int d\mu_N(Z_0) \int d\delta\, \psi_N(Z_0,\delta)\, \frac{1}{\delta_N\, N^2} \sum_i \sum_{j \in \mathcal{C}_i \cup \mathcal{C}_i^\delta} |K(X_i^\delta - X_j^\delta)|, \\ S_2 &= \int d\mu_N(Z_0) \int d\delta\, \psi_N(Z_0,\delta)\, \frac{1}{N^2} \sum_i \sum_{j \notin (\mathcal{C}_i \cup \mathcal{C}_i^\delta)} \frac{|K(X_i - X_j) - K(X_i^\delta - X_j^\delta)|}{\delta_N + \frac{1}{N}\sum_i |X_i - X_i^\delta|}. \end{aligned} \tag{4.2}$$

4.1 Bound on S_1

Recalling the bounds on $K = -\nabla\phi$ from Th. 1.2, one simply begins with a discrete Hölder inequality for any $\gamma \le 3$

$$S_1 \le \left(\int d\mu_N(Z_0) \int d\delta\, \psi_N(Z_0,\delta)\, \frac{1}{\delta_N\, N^2} \sum_i \sum_{j\in\mathcal{C}_i\cup\mathcal{C}_i^\delta} 1 \right)^{1-\alpha/\gamma}$$
$$\times \left(\int d\mu_N(Z_0) \int d\delta\, \psi_N(Z_0,\delta)\, \frac{1}{\delta_N\, N^2} \sum_i \sum_{j\neq i} \frac{1}{|X_i - X_j|^\gamma} \right)^{\alpha/\gamma}.$$

We first use the fact that the integral of ψ_N in δ is equal to 1 to get

$$\int d\mu_N(Z_0) \int d\delta\, \psi_N(Z_0,\delta)\, \frac{1}{\delta_N\, N^2} \sum_i \sum_{j\neq i} \frac{1}{|X_i - X_j|^\gamma} = \int d\mu_N(Z_0)\, \frac{1}{\delta_N\, N^2} \sum_i \sum_{j\neq i} \frac{1}{|X_i - X_j|^\gamma}.$$

We then perform the change of variable from Z_0 to Z, using the inverse flow of $Z(t, Z_0)$ which preserves the measure μ_N; one finds

$$\int d\mu_N(Z_0)\, \frac{1}{\delta_N\, N^2} \sum_i \sum_{j\neq i} \frac{1}{|X_i - X_j|^\gamma} \le \frac{1}{\delta_N\, N^2} \sum_i \sum_{j\neq i} \int d\mu_N(Z) \frac{1}{|X_i - X_j|^\gamma}.$$

As the second marginal of μ_N is bounded by c_β^2 by Lemma 3.2, this implies

$$\int d\mu_N(Z_0) \int d\delta\, \psi_N(Z_0,\delta)\, \frac{1}{\delta_N\, N^2} \sum_i \sum_{j\neq i} \frac{1}{|X_i - X_j|^\gamma} \le \frac{C\, c_\beta^2}{(3-\gamma)\delta_N}.$$

Hence

$$S_1 \le \frac{C\, c_\beta^{2\alpha/\gamma}}{(3-\gamma)\delta_N} \left(\int_{Z_0} d\mu_N(Z_0)\, \frac{1}{N^2} \sum_i (|\mathcal{C}_i| + |\mathcal{C}_i^\delta|) \right)^{1-\alpha/\gamma}.$$

To conclude, simply note that by definition $|\mathcal{C}_i^\delta| = |\mathcal{C}_i| = L$ and so for any $a > \frac{2\alpha}{3}$,

$$S_1 \le \frac{C_a\, c_\beta^a}{\delta_N} \left(\frac{L}{N} \right)^{1-\frac{a}{2}}, \tag{4.3}$$

where C_a satisfies $C_a \le \frac{C}{3a-2\alpha}$.

4.2 Bound on S_1^δ

Using Fubini and the change of variable $Z_0 \mapsto Z_0 + \delta$, and the image measure ν_N, we may rewrite

$$S_1^\delta = \int \nu_N(Z_0) dZ_0\, \frac{1}{\delta_N\, N^2} \sum_i \sum_{j\in\mathcal{C}_i\cup\mathcal{C}_i^{-\delta}} |K(X_i - X_j)|$$

And from the hypothesis (1.13), we may bound that that exactly as the previous one. The only difference is that a constant K_1 appears and that we shall use $\beta'(N)$ instead of β. We get

$$S_1^\delta \le \frac{CK_1\, c_{\beta'(N)}^{2\alpha/\gamma}}{\delta_N} \left(\frac{L}{N} \right)^{1-\alpha/\gamma}.$$

4.3 Bound on S_2

By using the assumption on the second derivative of ϕ in Th. 1.2, one first bounds

$$|K(X_i - X_j) - K(X_i^\delta - X_j^\delta)| \leq C\,(|X_i - X_i^\delta| + |X_j - X_j^\delta|)\,\left(\frac{1}{|X_i - X_j|^{\alpha+1}} + \frac{1}{|X_i^\delta - X_j^\delta|^{\alpha+1}}\right).$$

Therefore defining the following matrix, with $\mathbb{I}_A$ the characteristic function of the set A:

$$M_{ij} = \left(\frac{1}{|X_i - X_j|^{\alpha+1}} + \frac{1}{|X_i^\delta - X_j^\delta|^{\alpha+1}}\right)\,(\mathbb{I}_{j\notin(\mathcal{C}_i\cup\mathcal{C}_i^\delta)} + \mathbb{I}_{i\notin(\mathcal{C}_j\cup\mathcal{C}_j^\delta)}),$$

one has

$$S_2 \leq \int d\mu_N(Z_0) \int d\delta\,\psi_N(Z_0,\delta)\,\frac{\frac{1}{N^2}\sum_{i,j} M_{ij}|X_j - X_j^\delta|}{\delta_N + \frac{1}{N}\sum_k |X_k - X_k^\delta|}.$$

Consequently, if we use the classical matrix inequality $\|Mx\|_1 \leq \sup_j \left(\sum_i |M_{ij}|\right)\|x\|_1$,

$$S_2 \leq \int d\mu_N(Z_0) \int d\delta\,\psi_N(Z_0,\delta)\;\max_i \sum_j M_{ij}.$$

As before the terms in M containing $|X_i^\delta - X_j^\delta|$ are the equivalent of the ones with $|X_i - X_j|$, thanks to (1.13). Hence one has to bound

$$S_2 \leq C\,(S_2^1 + S_2^2),$$

with

$$S_2^1 = \int d\mu_N(Z_0) \int d\delta\,\psi_N(Z_0,\delta)\;\max_i \sum_{j\notin\mathcal{C}_i} \frac{1}{|X_i - X_j|^{\alpha+1}},$$

$$S_2^2 = \int d\mu_N(Z_0) \int d\delta\,\psi_N(Z_0,\delta)\;\max_i \sum_{j\ s.t.\ i\notin\mathcal{C}_j} \frac{1}{|X_i - X_j|^{\alpha+1}}.$$

Since nothing depends on δ now, one may integrate $\psi_N(Z_0,\delta)$ in δ with value 1. Moreover changing variable from Z_0 to Z (we recall the flow is measure preserving), one simply finds

$$S_2^1 = \int d\mu_N(Z)\;\max_i \sum_{j\notin\mathcal{C}_i} \frac{1}{|X_i - X_j|^{\alpha+1}},$$

$$S_2^2 = \int d\mu_N(Z)\;\max_i \sum_{j\ s.t.\ i\notin\mathcal{C}_j} \frac{1}{|X_i - X_j|^{\alpha+1}}.$$

Let us now carefully bound each of these terms.

The S_2^1 term

We use

$$\int f(X)\,d\mu_N = \int_0^{+\infty} P(f(X) > l)\,dl\ , \tag{4.4}$$

where P is the probability with respect to the measure μ_N on $\Pi^{3N}\times\mathbb{R}^{3N}$. We have to evaluate now expressions like

$$P\left(\max_i \frac{1}{N}\left(\sum_{j\notin\mathcal{C}_i} \frac{1}{|X_i - X_j|^{\alpha+1}}\right) > l\right) = P\left(\exists i\ s.t.\ \frac{1}{N}\left(\sum_{j\notin\mathcal{C}_i} \frac{1}{|X_i - X_j|^{\alpha+1}}\right) > l\right)\ . \tag{4.5}$$

To bound this probability, we will need the following lemma:

Lemma 4.1 *Given $x_1, \ldots, x_n \geq 0$ and $l \geq 0$; given $(u_k)_{k=1}^n$ such that $\sum_{k=1}^n u_k = 1$. If $\sum_{i=1}^n x_i > l$, then $\exists k \in [1,\ n],\ \exists i_1, \ldots, i_k \ /\ x_{i_r} > l u_k\ ,\ \forall r = 1, \ldots, k$.*

Proof: Let the x_i be sorted $x_1 \geq x_2 \geq \ldots \geq x_n$, and suppose the conclusion is not true. Then we have

$$x_1 \leq l u_1, \ldots, x_n \leq l u_n\ .$$

Thus $\sum_i x_i \leq l \sum_i u_k = l$. □

We apply the lemma to Eq. 4.5, with $u_k = \frac{C_N \nu}{k} \left(\frac{k}{N}\right)^\nu$ and $0 < \nu < 1$ to be determined. C_N is chosen such that $\sum_{k=1}^N u_k = 1$. Using Riemann sums, we see that $\lim_{+\infty} C_N = 1$, and that $C_N \geq 1, \forall N \geq 1$. Hence we get

$$\begin{aligned}
P_l &= P\left(\exists i\ s.t.\ \frac{1}{N}\left(\sum_{j \notin \mathcal{C}_i} \frac{1}{|X_i - X_j|^{\alpha+1}}\right) > l\right) \\
&\leq \sum_{k=1}^{N-L} P\left(\exists i; j_1, \ldots, j_k \notin \mathcal{C}_i,\ \frac{1}{N}\frac{1}{|X_i - X_{j_r}|^{\alpha+1}} > l\frac{C_N \nu}{k}(k/N)^\nu\right) \\
&\leq \sum_{k=1}^{N-L} P\left(\exists i; j_1, \ldots, j_k \notin \mathcal{C}_i,\ |X_i - X_{j_r}| < \left(\frac{1}{\nu l}\right)^{1/(\alpha+1)} \left(\frac{k}{N}\right)^{\lambda/3}\right) \\
&= \sum_{k=1}^{N-L} P_{l,k}\ ,
\end{aligned}$$

where for simplicity we have introduced the parameter $\lambda = 3(1-\nu)/(\alpha+1)$.

To estimate the probability $P_{l,k}$, once the particle i is chosen, we have a constraint on the position of k particles, which have to be close enough to particle i, plus constraints on the position of L distinct particles, from the definition of $\mathcal{C}_i$. This event concern $k+L+1$ particles, and to estimate it, we will use an estimate of its volume $P^u_{l,k}$ in the configuration space $\mathbb{T}^{3(k+L+1)} \times \mathbb{R}^{3(k+L+1)}$. It thus involves the $(k+L+1)$ marginal of μ_N which is bounded by c_β^{k+L+1} by Lemma 3.2.

This leads to the following estimates

$$P_{l,k} \leq C\ c_\beta^{k+L+1}\, N\ \left(\frac{1}{\nu l}\right)^{\frac{3(k+L)}{\alpha+1}} \left(\frac{k}{N}\right)^{\lambda(k+L)}\ .$$

Moreover, using a simplified version of Binet formula (See [25])

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{\theta}{12n}}\ ,\ \text{for some } \theta \in (0,1)$$

the binomial coefficient C_N^p may be bound by:

$$\begin{aligned}
C_N^p &= \frac{N!}{p!(N-p)!} = (2\pi)^{-\frac{1}{2}} e^{\frac{\theta_N}{12N} - \frac{\theta_p}{12p} - \frac{\theta_{N-p}}{12(N-p)}} \frac{N^N}{p^p (N-p)^{N-p}} \left(\frac{N}{p(N-p)}\right)^{1/2} && (4.6) \\
&\leq (2\pi)^{-\frac{1}{2}} e^{\frac{1}{12N}} \left(\frac{N}{N-1}\right)^{1/2} \left(\frac{N}{p}\right)^p \left(1 + \frac{p}{N-p}\right)^{N-p} \leq \left(\frac{Ne}{p}\right)^p\ . && (4.7)
\end{aligned}$$

And we do not forget that since $C_N^p = C_N^{N-p}$ we may use the same inequality with p replaced by $N-p$. Inserting this in the above inequality, we get:

$$\begin{aligned}
P_{l,k} &\leq C c_\beta^{k+L+1} N \left(\frac{N}{k+L}\right)^{k+L} e^{k+L} \left(\frac{1}{\nu l}\right)^{\frac{3(k+L)}{\alpha+1}} \left(\frac{k}{N}\right)^{\lambda(k+L)} \\
&\leq C c_\beta\, N \left(\frac{k}{N}\right)^{(\lambda-1)(k+L)} \left(\frac{A c'_\beta}{\nu l}\right)^{\frac{3(k+L)}{\alpha+1}} \\
&\leq C c_\beta \left(\frac{k}{N}\right)^{(\lambda-1)(k+L)-1} k \left(\frac{A c'_\beta}{\nu l}\right)^{\frac{3(k+L)}{\alpha+1}}\ ,
\end{aligned}$$

where $c'_\beta = c_\beta^{(\alpha+1)/3}$, and $A = e^{(\alpha+1)/3}$ is a numerical constant. Now taking ν close enough to 0 (precisely $\nu < \frac{2-\alpha}{3}$), one has $\lambda > 1$ and then we take as well $L \geq (\lambda-1)^{-1}$ (recall that L has yet to be fixed) ; hence

$$P_{l,k} \leq C c_\beta \, k \left(\frac{Ac'_\beta}{\nu l}\right)^{\frac{3(k+L)}{\alpha+1}} .$$

If we sum on k, we get:

$$\begin{aligned} \sum_{k=0}^{N-L} P_{l,k} &\leq C c_\beta \left(\frac{Ac'_\beta}{\nu l}\right)^{\frac{3L}{\alpha+1}} \sum_{k=0}^{N-L} k \left(\frac{Ac'_\beta}{\nu l}\right)^{\frac{3k}{\alpha+1}} &(4.8)\\ &\leq C\, c_\beta \left(\frac{Ac'_\beta}{\nu l}\right)^{\frac{3(L+1)}{\alpha+1}} \frac{1}{(1-(Ac'_\beta/\nu l)^{3/(\alpha+1)})^2} \leq C\ c_\beta \left(\frac{Ac'_\beta}{\nu l}\right)^{\frac{3(L+1)}{\alpha+1}} , &(4.9)\end{aligned}$$

provided $l \geq l_0 = \frac{2\,A\,c'_\beta}{\nu}$. If we take moreover $L \geq p \geq p(\alpha+1)/3$ for some $p \geq 0$, we get a simpler bound:

$$\sum_{k=0}^{N-L} P_{l,k} \leq C\, c_\beta \left(\frac{A\,c'_\beta}{\nu l}\right)^p .$$

Remark that the conditions on L depend only on α and λ (which depends only on α). In particular, those conditions are independent of the parameter β. Thus, we have

$$\begin{aligned} P\left(\exists i\ s.t.\ \frac{1}{N}\left(\sum_{j\notin\mathcal{C}_i} \frac{1}{|X_i - X_j|^{\alpha+1}}\right) > l\right) &\leq 1 \text{ for } l < l_0 = \frac{2A}{\nu} c_\beta^{\frac{\alpha+1}{3}} &(4.10)\\ &\leq C c_\beta^{1+p\frac{\alpha+1}{3}} (\nu l)^{-p} \text{ for } l \geq l'_0 &(4.11)\end{aligned}$$

Integrating this quantity in l, one obtains for M such that $C c_\beta^{1+p\frac{\alpha+1}{3}}(\nu M)^{-p} = 1$ that

$$\begin{aligned} \int_0^\infty P(\cdots > l)\, dl &= \int_0^M P(\cdots > l)\, dl + \int_M^\infty P(\cdots > l)\, dl \\ &\leq M + C c_\beta \left(\frac{Ac'_\beta}{\nu}\right)^p \int_M^\infty \frac{dl}{l^p} \\ &\leq M + C c_\beta \left(\frac{Ac'_\beta}{\nu}\right)^p \frac{M^{1-p}}{p-1} \\ &= \frac{p}{p-1} M \leq \frac{C}{\nu}\ c_\beta^{\frac{1}{p}+\frac{\alpha+1}{3}} \end{aligned}$$

for $p \geq \frac{3}{2-\alpha}$. And finally, we get that

$$S_2^1 \leq \frac{C}{\nu}\ c_\beta\ ,$$

for some constant C, provided that L is large enough.

Since every calculation have been performed, we see that a possible for ν is $\nu = \frac{2-\alpha}{6}$ in which case the condition on L is exactly

$$L \geq \frac{6}{2-\alpha}$$

With that choice of ν, we get

$$S_2^1 \leq \frac{C}{2-\alpha}\ c_\beta\ ,$$

The S_2^2 term: Through the same type of computations, we are led to evaluate expressions like

$$
\begin{aligned}
P'_l &= P\left(\exists j \; s.t. \; \frac{1}{N}\left(\sum_{i/j\notin\mathcal{C}_i}\frac{1}{|X_i-X_j|^{\alpha+1}}\right) > l\right) && (4.12)\\
&\le \sum_{k=1}^{N} P\left(\exists j; i_1,\ldots,i_k \;/\; j\notin\mathcal{C}_{i_r}\frac{1}{N}\frac{1}{|X_j-X_{i_r}|^{\alpha+1}} > l\frac{C_N\nu}{k}\left(\frac{k}{N}\right)^{\nu}\right) && (4.13)\\
&\le \sum_{k=1}^{N} P\left(\exists j; i_1,\ldots,i_k \;/\; j\notin\mathcal{C}_{i_r}|X_j-X_{i_r}| < \frac{1}{(\nu l)^{\alpha+1}}\left(\frac{k}{N}\right)^{(1-\nu)/(\alpha+1)}\right) && (4.14)\\
&= \sum_{k=1}^{N} P'_{l,k}\;. && (4.15)
\end{aligned}
$$

Since the sum is performed on the particles i such that $j\notin\mathcal{C}_i$, we cannot choose $L+k$ particles close to j as for S_2^1. But, we have nevertheless to choose k particles close enough to j, a probability that will give a good bound if $k\ge L$. If $k\le L$, once a particle i close to j such that $j\notin\mathcal{C}_i$ is chosen, one knows that there exist L other particles close to j. This will be enough to bound the probability.

In the second case ($k\le L$), we pick up a particle j (N possibilities), at least another particle i (since $k\ge 1$) and then we have to choose L other particles closer to i than j is. Since $|X_j-X_i|$ has to be less than $C\,l^{-1/(\alpha+1)}\,(k/N)^{(1-\nu)/(\alpha+1)}$,

$$
\begin{aligned}
\sum_{k=0}^{L} P'_{k,l} &\le \sum_{k=0}^{L} c_\beta^{L+2}N^2C_N^L\left(\frac{1}{\nu l}\right)^{3L/(\alpha+1)}\left(\frac{k}{N}\right)^{\lambda L} && (4.16)\\
&\le C\sum_{k=0}^{L} c_\beta^2\,N^2\left(\frac{L}{N}\right)^{(\lambda-1)L}\left(\frac{Ac'_\beta}{\nu l}\right)^{3L/(\alpha+1)} && (4.17)\\
&\le Cc_\beta^2\,N^3\left(\frac{L}{N}\right)^{(\lambda-1)L}\left(\frac{Ac'_\beta}{\nu l}\right)^{3L/(\alpha+1)}\;. && (4.18)
\end{aligned}
$$

If $L\le\sqrt{N}$, we may use

$$
N^3\left(\frac{L}{N}\right)^{(\lambda-1)L} \le N^{3-\frac{\lambda-1}{2}L} \le 1\,,
$$

as soon as $L\ge\frac{6}{\lambda-1}$. In that case, we may use this bound in last inequality and obtain:

$$
\begin{aligned}
\sum_{k=0}^{L} P'_{k,l} &\le Cc_\beta^2\left(\frac{Ac'_\beta}{l}\right)^{3L/(\alpha+1)}\\
&\le Cc_\beta^2\left(\frac{Ac'_\beta}{l}\right)^{3L/(\alpha+1)}\;,
\end{aligned}
$$

if $L\le\sqrt{N}$.

In the first case $k>L$, we pick up the particle j, and then choose k particles i_r close to j. We obtain as previously

$$\begin{aligned}
\sum_{k=L+1}^{N-1} P'_{k,l} &\leq \sum_{k=L+1}^{N} c_\beta^{k+1} N C_N^k \left(\frac{1}{\nu l}\right)^{3k/(\alpha+1)} \left(\frac{k}{N}\right)^{\lambda k} \\
&\leq C \sum_{k=L+1}^{N} c_\beta\, N \left(\frac{k}{N}\right)^{(\lambda-1)k} \left(\frac{Ac'_\beta}{\nu l}\right)^{3k/(\alpha+1)} \\
&= C\, c_\beta \sum_{k=L+1}^{N} k \left(\frac{k}{N}\right)^{(\lambda-1)k-1} \left(\frac{Ac'_\beta}{l}\right)^{3k/(\alpha+1)} \\
&\leq C c_\beta \sum_{k=L+1}^{N} k \left(\frac{Ac'_\beta}{\nu l}\right)^{3k/(\alpha+1)} \\
&\leq C c_\beta \left(\frac{Ac'_\beta}{\nu l}\right)^{3L/(\alpha+1)} \qquad ,
\end{aligned}$$

where we again restricted ourselves to $(\lambda-1)L \geq 1$ and assume $l \geq l_0$. Putting the two sum together, we get the bound

$$\sum_{k=1}^{N-1} P'_{k,l} \leq C c_\beta^2 \left(\frac{Ac'_\beta}{\nu l}\right)^{3L/(\alpha+1)}$$

It remains to integrate in l. Doing exactly as for the S_2^1 term, and choosing the same ν, we will get

$$S_2^2 \leq \frac{C}{2-\alpha}\, c_\beta.$$

The only difference is that it will require $p \geq \frac{6}{2-\alpha}$ and thus $L \geq \frac{12}{2-\alpha}$

4.4 Conclusion of the proof

Putting all together, we finally we may bound

$$\frac{dQ}{dt} \leq 1 + C_a \frac{c_\beta^a}{\delta_N} \left(\frac{L}{N}\right)^{1-\frac{a}{2}} + \frac{C}{2-\alpha}\, c_\beta,$$

with $c_\beta = e^{-\beta\phi_{min}}$ and $a > \frac{2\alpha}{3}$, where L is subject to the restrictions (with the choice $\nu = \frac{2-\alpha}{6}$ which means that $\lambda = \frac{4+\alpha}{2+2\alpha}$)

$$L \geq \frac{36}{2-\alpha}, \quad , \quad L \leq \sqrt{N}.$$

It is possible only if $N \geq \frac{6^4}{(2-\alpha)^2}$ and in that case it is clear that L should be chosen as small as possible and from the constraint that means

$$L = \frac{36}{2-\alpha}.$$

With this choice, one has

$$\frac{dQ}{dt} \leq 1 + \frac{C_a}{2-\alpha}\, c_\beta^a \frac{1}{\delta_N N^{1-\frac{a}{2}}} + \frac{C}{2-\alpha}\, c_\beta.$$

Now if one takes $\delta_N = N^{-\varepsilon}$, we can get a uniform bound in N, only if

$$\varepsilon \leq 1 - \frac{a}{2}.$$

If this is true, we get

$$\frac{dQ}{dt} \leq 1 + \frac{C_a\, c_\beta^a \; + C\, c_\beta}{2-\alpha}.$$

with $C_a \leq \frac{C}{3a-2\alpha}$ which is the result given by Theorem 1.2.

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